The Field Equations of Metric-Affine Gravitational Theories

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The notions of conservation of charge and dimensional consistency are used to obtain conditions which uniquely characterize the field equations of electromagnetism and gravitation in a metric-affine gravitational framework with a vector potential. Conditions for the uniqueness of the choice of field equations of a metric-affine gravitational theory (in the absence of electromagnetism) follow as a special case. Some consequences are discussed.

§ 1. Introduction

By introducing the concept of hypermomentum, Hehl, Kerlick, and von der Heyde [1] have been led to consider the possibility of a metric-affine framework for a gravitational theory. The mathematical basis for this framework is a four-dimensional (differentiable) manifold possessing a Lorentzian metric g (components g_{ij}) and a linear connection ∇ (components $\Gamma_j{}^i{}_k$). The linear connection is not assumed to be symmetric, nor is it assumed that $\nabla g = 0$. In this paper we include a vector field (with components ψ_i) into our analysis to describe electromagnetic phenomena [2]. The vector field, the metric and the linear connection are assumed to be specified through field equations which take the form

$$A^{ij} = \varkappa \sqrt{g}\, T^{ij}, \tag{1.1}$$

$$C^{lm}{}_k = \varkappa \sqrt{g}\, \Delta_k{}^{ml}, \tag{1.2}$$

and

$$B^i = \varkappa' \sqrt{g}\, J^i \tag{1.3}$$

where $\varkappa$ and $\varkappa'$ are constants constructed from c (the speed of light) and K (the gravitational constant) (e.g. $\varkappa = 8\pi K/c^4$ and $\varkappa' = 16\pi/c$). In Eqs. (1.1)–(1.3), $g := |\det(g_{ab})|$ and A^{ij}, $C^{lm}{}_k$, and B^i are tensor density concomitants of the metric, the difference tensor [3] $W_{ij}{}^k := \Gamma_i{}^k{}_j - \{_i{}^k{}_j\}$ ($\{_i{}^k{}_j\}$ are the Christoffel symbols of the second kind built from g_{ab}), the vector field ψ_i and their derivatives. The „source" terms T^{ij}, $\Delta_k{}^{ml}$, and J^i are the energy-momentum tensor, the hypermomentum tensor [1] and the charge-current vector (respectively) of all non-electromagnetic matter fields.

The primary concern of the present article is the choice one should take for A^{ij}, $C^{lm}{}_k$ and B^i in (1.1)–(1.3). Here we shall demand that there exist a Lagrangian $\mathscr{L}$ for which $A^{ij} = \delta\mathscr{L}/\delta g_{ij}$, $C^{lm}{}_k = \frac{1}{2}\delta\mathscr{L}/\delta W_{lm}{}^k$, and $B^i = \delta\mathscr{L}/\delta\psi_i$. Another restriction on the concomitant B^i follows from the concept of charge conservation which takes the form $(\sqrt{g}\, J^i)_{,i} = 0$. Hence it is reasonable to require that $B^i{}_{,i} = 0$ identically.

Even with these assumptions there is a large degree of indeterminancy in the choices for A^{ij}, $C^{lm}{}_k$ and B^i. For example, there are (locally at least) 58 independent (scalar) invariants of the metric and the difference tensor (no derivatives) [4]. If $\{I_\alpha\}$ is the set of these invariants, then any Lagrangian for A^{ij} and $C^{lm}{}_k$ (and B^i) could be augmented by adding $\sqrt{g}\, f(I_\alpha)$ where f is an arbitrary differentiable function of 58 variables. All the above conditions would still be satisfied. Obviously some further assumptions on the structure of the concomitants A^{ij}, $C^{lm}{}_k$ and B^i must be made before any consequences of practical value may be obtained. In the next section an assumption is introduced which allows us to resolve this problem.

§ 2. A Theorem on the Field Equations in Metric-Affine Theories

To simplify the problem discussed above, one further assumption will be introduced. We base this assumption on the concept of dimensional consistency which has led to considerable simplification in several concomitant problems involving relativistic gravitational theories [5]. Choosing units so that $c = K = 1$ (with $x^i \sim L^1$) we have $T^{ij} \sim L^{-2}$, $J^i \sim L^{-2}$ and $\Delta_k{}^{ml} \sim L^{-1}$. As in previous work, $g_{ij,k_1\ldots k_\alpha} \sim L^{-\alpha}$ ($\alpha = 0, 1, \ldots$) and $\psi_{i,j_1\ldots j_\beta} \sim L^{-\beta}$ ($\beta = 0, 1, \ldots$). The difference tensor $W_{ij}{}^k$ must have the same units as $\{_i{}^k{}_j\}$ ($\sim L^{-1}$) so we have $W_{ij}{}^k{}_{,l_1\ldots l_\gamma} \sim L^{-\gamma-1}$ ($\gamma = 0, 1, \ldots$). With the units

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given as above we impose the additional assumption that the *concomitants* $A^{ij}, C^{lm}{}_k$, *and* B^i *do not involve any constants with non-zero dimensions.* If A^{ij}, $C^{lm}{}_k$ and B^i are to enter into field equations of the form (1.1)—(1.3) then $A^{ij} \sim L^{-2}$, $B^i \sim L^{-2}$ and $C^{lm}{}_k \sim L^{-1}$. Our main result is the following

Theorem: *Suppose that* A^{ij} *(of class* C^3*),* B^i *(of class* C^3*) and* $C^{lm}{}_k$ *(of class* C^2*), are tensor density concomitants of* g_{ab}, ψ_r, $W_{rs}{}^t$ *and their various partial derivatives to some finite order. If they satisfy the following conditions:*

(a) $A^{ij} \sim L^{-2}$, $B^i \sim L^{-2}$, $C^{lm}{}_k \sim L^{-1}$, *they satisfy the axiom of dimensional analysis, and do not involve constants with non-zero dimension;*

(b) *There exists a function* $\mathcal{L}$ *which is a concomitant of* g_{ab}, ψ_r, $W_{rs}{}^t$ *and their various partial derivatives to some finite order such that*

$$A^{ij} = \frac{\delta \mathcal{L}}{\delta g_{ij}}, \quad B^i = \frac{\delta \mathcal{L}}{\delta \psi_i}, \quad \text{and} \quad C^{lm}{}_k = \frac{1}{2} \frac{\delta \mathcal{L}}{\delta W_{lm}{}^k};$$

(c) $B^i{}_{,i} = 0$; *then in a four-space,*

$$A^{ij} = a_1 \sqrt{g}\, G^{ij} - \frac{a_2 \sqrt{g}}{2} (F^i{}_l F^{jl} - \tfrac{1}{4} g^{ij} F^{rs} F_{rs}) + \tfrac{1}{2} g^{ij} C^{lm}{}_k W_{lm}{}^k + C^{mlj} W_{ml}{}^i - C^{iml} W^j{}_{ml} - C^{mi}{}_l W_m{}^{jl}, \quad (2.1)$$

$$B^i = a_2 \sqrt{g}\, F^{ij}|_j, \quad (2.2)$$

and

$$\begin{aligned} C^{lm}{}_k = \sqrt{g}\, \{ & b_1 g^{lm} W_{kr}{}^r + b_1 \delta^m_k W_r{}^{rl} + b_2 g^{lm} W_{rk}{}^r \\ & + b_2 \delta^l_k W_r{}^{rm} + b_3 \delta^l_k W^{mr}{}_r + b_3 \delta^m_k W_r{}^{lr} \\ & + b_4 W_k{}^{lm} + b_4 W^m{}_k{}^l + b_5 g^{lm} W^r{}_{rk} \\ & + b_6 \delta^l_k W_r{}^{mr} + b_7 \delta^m_k W^{lr}{}_r + b_8 W^{lm}{}_k \\ & + b_9 W^l{}_k{}^m + b_{10} W^{ml}{}_k + b_{11} W_k{}^{ml} \} \\ & + c_1 g^{lm} g_{ku} \varepsilon^{urst} W_{rst} + c_1 W^r{}_{rt} g_{sk} \varepsilon^{tlms} \\ & + c_2 \delta^l_k \varepsilon^{mrst} W_{rst} + c_2 W_{rt}{}^r g_{sk} \varepsilon^{tlms} \\ & + c_3 W_r{}^l{}_t g_{ks} \varepsilon^{srmt} + c_3 W^m{}_{rt} g_{sk} \varepsilon^{tlrs} \\ & + c_4 W_{rt}{}^l g_{sk} \varepsilon^{srtm} + c_4 W_{krt} \varepsilon^{tlmr} \\ & + c_5 \delta^m_k \varepsilon^{lrst} W_{rst} + c_5 W_{tr}{}^r g_{sk} \varepsilon^{tlms} \\ & + c_6 W_{rt}{}^m g_{ks} \varepsilon^{srtl} + c_6 W_{rkt} \varepsilon^{tlmr}, \end{aligned} \quad (2.3)$$

where $F_{ij} := \psi_{j,i} - \psi_{i,j}$, a_1, a_2, b_α $(\alpha = 1, \ldots, 11)$, *and* c_β $(\beta = 1, \ldots, 6)$ *are arbitrary constants. A Lagrangian which satisfies condition* (b) *is given by*

$$\mathcal{L} = -a_1 \sqrt{g}\, R + C^{rs}{}_r W_{rs}{}^t + \frac{a_2}{4} \sqrt{g}\, F^{rs} F_{rs}. \quad (2.4)$$

(Note that $\mathcal{L}$ *is a scalar density although* (b) *does not require this.)*

The proof of this theorem is in the next section.

If one accepts the assumptions of the theorem as physically reasonable, then regardless of the arbitrariness of the constants in (2.1)—(2.3), there are some quite general conclusions that follow. Equation (1.2) with $C^{lm}{}_k$ given by (2.3) is a linear algebraic (i.e. non-differential) equation for $W_{ij}{}^k$. A reasonable requirement is to demand that one can always solve (1.2) with (2.3) uniquely for $W_{ij}{}^k$ in terms of $\Delta^{lm}{}_k$ and g_{ab}. A sufficient condition for this is given in the following:

Lemma: *Suppose* $C^{lm}{}_k$ *and* $W_{rs}{}^t$ *are related by Equation* (2.3). *If*

$$(2 b_4 + b_8 + b_9 + b_{10} + b_{11}) \det(M_1) \det(M_2) \neq 0,$$

where

$$M_1 = \begin{bmatrix} b_1 + b_3 + 4 b_7 + b_8 + b_9 & 4 b_1 + b_2 + b_4 + b_5 + b_{11} & b_2 + 4 b_3 + b_4 + b_6 + b_{10} & c_1 + c_2 - c_4 + 4 c_5 + c_3 \\ b_1 + 4 b_3 + b_4 + b_7 + b_{10} & b_1 + 4 b_2 + b_4 + b_5 + b_9 & b_2 + b_3 + 4 b_6 + b_8 + b_{11} & c_1 + 4 c_2 + c_5 - c_3 - c_6 \\ 4 b_1 + b_3 + b_4 + b_7 + b_{11} & b_1 + b_2 + 4 b_5 + b_8 + b_{10} & 4 b_2 + b_3 + b_4 + b_6 + b_9 & 4 c_1 + c_2 + c_4 + c_5 + c_6 \\ 6 c_5 + 2 c_3 - 2 c_4 & 6 c_1 + 2 c_4 + 2 c_6 & 6 c_2 - 2 c_3 - 2 c_6 & b_9 + b_{10} + b_{11} - b_8 - 2 b_4 \end{bmatrix},$$

and

$$M_2 = \begin{bmatrix} b_4 + b_9 - b_8 - b_{11} & b_4 + b_{10} - b_8 - b_{11} & c_4 - c_3 - c_6 & -c_3 - c_4 - c_6 \\ b_8 + b_{10} - b_4 - b_9 & b_{11} - b_9 & c_3 - c_4 - c_6 & 2 c_3 \\ 5 c_3 + c_6 - c_4 & c_4 + 5 c_6 + c_3 & b_4 + b_{11} - b_8 - b_9 & b_4 + b_{11} - b_{10} - b_8 \\ c_6 - c_3 + 5 c_4 & 4 c_4 - 2 c_3 - 4 c_6 & b_8 + b_9 - b_4 - b_{10} & b_9 - b_{11} \end{bmatrix}$$

then $C^{lm}{}_k = 0$ *implies that* $W_{ij}{}^k = 0$, *i.e. there exists a unique solution for* $W_{ij}{}^k$ *in terms of* $C^{lm}{}_k$ *(and* g_{ab}*).* □

(The proof is straightforward and is available from the author upon request.)

Under the conditions of the lemma it is clear that if $\Delta_k{}^{ml} = 0$, then via (1.2), $C^{lm}{}_k = 0$ and we must have $W_{ij}{}^k = 0$. Thus in vacuum [$T^{ij} = 0$, $J^i = 0$, and $\Delta_k{}^{ml} = 0$ in (1.1)—(1.3)], the field equations which follow from the theorem reduce to the vacuum Einstein-Maxwell equations. Evidently (under our assumptions) there is no propagation of non-Riemannian effects ($W_{ij}{}^k \neq 0$) in vacuum.

Furthermore, photons will not produce non-Riemannian effects under the above assumptions. (Thus for photons $\Delta^{lm}{}_k=0$ and so $W_{ij}{}^k=0$. The latter comment is the metric-affine theory conterpart to the result that photons do not produce torsion in the Einstein-Cartan theory.) Another observation is that any metric-affine theory whose field equations satisfy the above assumptions must imply general relativity for vacuum.

Any generalization of the above results will likely require specific assumptions regarding the nature of the dependence of the concomitants A^{ij}, B^i and $C^{lm}{}_k$ upon constants with non-zero dimensions. One possibility is to assume, for example, that

$$C^{lm}{}_k = C_1{}^{lm}{}_k + \lambda C_2{}^{lm}{}_k + \lambda^2 C_3{}^{lm}{}_k + \cdots + \lambda^{\beta-1} C_\beta{}^{lm}{}_k$$

where the constant $\lambda \sim L^1$ and the $C_\alpha{}^{lm}{}_k, \alpha=1,\ldots,\beta$ are each tensor densities with $C_\alpha{}^{lm}{}_k \sim L^{-\alpha}$ for $\alpha=1,\ldots,\beta$. The techniques of the present paper can then be applied to each of the tensor densities $C_\alpha{}^{lm}{}_k$. While this method is systematic and in principle quite trivial, in practice it very quickly leads to unwieldy expressions [6].

§ 3. Proof of the Theorem

The proof is essentially the same as that presented in Ref. [2], the differences being based upon the fact that $W_{ab}{}^c$ has no symmetries whereas the torsion satisfies $S_{ab}{}^c = -S_{ba}{}^c$. Proposition (1) of Ref. [2] with $S_{ab}{}^c$ replaced by $W_{ab}{}^c$ is valid in the present situation as is Proposition (2) (Ref. [2]). Thus A^{ij} and B^i are linearly homogeneous in $g_{ab,cd}$, $W_{ab}{}^c{}_{,d}$ and $\psi_{a,bc}$ and quadratically homogeneous in $g_{ab,c}$, $\psi_{a,b}$ and $W_{ab}{}^c$ while $C^{lm}{}_k$ is linearly homogeneous in the latter set. All these terms have coefficients which depend upon g_{ab} only. From the proof of Proposition (3) (Ref. [1]) it follows that

$$C^{lm}{}_k = \eta^{lm}{}_k{}^{rs}{}_t(g_{ab})\, W_{rs}{}^t\,, \tag{3.1}$$

where $\eta{:::}$ is a tensor density concomitant of the indicated form. Using Anderson's theorem [7] we find that $\eta^{lm}{}_k{}^{rs}{}_t = \eta^{rs}{}_t{}^{lm}{}_k$. From Weyl [8] (page 52, noting page 65), we conclude that

$$\begin{aligned}\eta^{lmkrst} = \sqrt{g}\,\{&b_1(g^{lm}g^{kr}g^{st}+g^{lt}g^{mk}g^{rs})\\ &+b_2(g^{lm}g^{ks}g^{rt}+g^{lk}g^{mt}g^{rs})\\ &+b_3(g^{lk}g^{mr}g^{st}+g^{ls}g^{km}g^{rt})\\ &+b_4(g^{ls}g^{kr}g^{mt}+g^{lt}g^{mr}g^{ks})\\ &+b_5\,g^{lm}g^{kt}g^{rs}+b_6\,g^{lk}g^{ms}g^{rt}\\ &+b_7\,g^{lr}g^{mk}g^{st}+b_8\,g^{lr}g^{ms}g^{kt}\\ &+b_9\,g^{lr}g^{mt}g^{ks}+b_{10}\,g^{ls}g^{kt}g^{mr}\\ &+b_{11}\,g^{lt}g^{ms}g^{kr}\}\\ &+c_1(g^{lm}\varepsilon^{krst}+g^{rs}\varepsilon^{tlmk})\\ &+c_2(g^{lk}\varepsilon^{mrst}+g^{rt}\varepsilon^{slmk})\\ &+c_3(g^{ls}\varepsilon^{krmt}+g^{rm}\varepsilon^{tlsk})\\ &+c_4(g^{lt}\varepsilon^{krsm}+g^{rk}\varepsilon^{tlms})\\ &+c_5(g^{mk}\varepsilon^{lrst}+g^{st}\varepsilon^{rlmk})\\ &+c_6(g^{mt}\varepsilon^{krsl}+g^{sk}\varepsilon^{tlmr})\,,\end{aligned} \tag{3.2}$$

where b_α, $\alpha=1,\ldots,11$ and c_β, $\beta=1,\ldots,6$ are arbitrary constants. To obtain Eq. (3.2) we have employed the fact that $2\eta^{lmkrst}=\eta^{lmkrst}+\eta^{rstlmk}$ along with the identity

$$\begin{aligned}2g^{rl}\varepsilon^{kmst} = &-(g^{lt}\varepsilon^{krsm}+g^{rk}\varepsilon^{tlms})\\ &-(g^{ls}\varepsilon^{krmt}+g^{rm}\varepsilon^{tlsk})\\ &-(g^{lm}\varepsilon^{rkst}+g^{rs}\varepsilon^{ltmk})\\ &-(g^{lk}\varepsilon^{mrst}+g^{rt}\varepsilon^{slmk})\end{aligned}$$

which follows from the fact that in a 4-space

$$\delta^{lkmst}_{abcde}\,g^{ra}\,\varepsilon^{bcde} \equiv 0\,.$$

Substitution from Eq. (3.2) into Eq. (3.1) then yields Equation (2.3).

The proof now follows the same steps as in Ref. [2] with $S_{ab}{}^c$ replaced by $W_{ab}{}^c$. In this way we can establish Eq. (2.2) and show that

$$A^{ij} = a_1\sqrt{g}\,G^{ij} - \frac{a_2\sqrt{g}}{2}\,(F^i{}_l F^{jl} - \tfrac{1}{4} g^{ij} F^{rs}F_{rs}) + \zeta^{ijab}{}_c{}^{lm}{}_k\, W_{ab}{}^c W_{lm}{}^k \tag{3.3}$$

where $\zeta{:::}$ is a tensor density concomitant of g_{rs} only. Using Anderson's result again we find that

$$\frac{\partial}{\partial W_{lm}{}^k}\left(\frac{\delta\mathscr{L}}{\delta g_{ij}}\right) = \frac{\partial}{\partial g_{ij}}\left(\frac{\delta\mathscr{L}}{\delta W_{lm}{}^k}\right)$$

which implies that

$$2\,\zeta^{ijab}{}_c{}^{lm}{}_k\, W_{ab}{}^c = 2\,\frac{\partial C^{lm}{}_k}{\partial g_{ij}}\,. \tag{3.4}$$

Note that in taking the $\partial/\partial g_{ij}$ derivative in (3.4) we are holding $W_{ab}{}^c$ constant. Thus multiplying (3.4) by $W_{lm}{}^k$ we deduce that

$$\zeta^{ijab}{}_c{}^{lm}{}_k\, W_{ab}{}^c W_{lm}{}^k = \partial C^{lm}{}_k\, W_{lm}{}^k/\partial g_{ij}\,.$$

However, $C^{lm}{}_k W_{lm}{}^k$ is a scalar density concomitant of g_{ab} and $W_{ab}{}^c$ and thus from the invariance identity [9] we find that

$$\delta^s_r C^{lm}{}_k W_{lm}{}^k = 2 \frac{\partial C^{lm}{}_k W_{lm}{}^k}{\partial g_{si}} g_{ri} + 2 (C^{sm}{}_k W_{rm}{}^k + C^{ms}{}_k W_{mr}{}^k - C^{ml}{}_r W_{ml}{}^s).$$

Consequently we have

$$\zeta^{ijab}{}_c{}^{lm}{}_k W_{ab}{}^c W_{lm}{}^k = \tfrac{1}{2} g^{ij} C^{lm}{}_k W_{lm}{}^k + C^{mlj} W_{ml}{}^i - C^{imk} W^j{}_{mk} - C^{mi}{}_k W_m{}^{jk}$$

which coupled with (3.3) implies (2.1). To show that the Lagrangian (2.4) satisfies condition (b) of the theorem is straightforward.

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[1] F. W. Hehl, G. D. Kerlick, and P. von der Heyde, Z. Naturforsch. **31a**, 111, 524, and 823 (1976); Phys. Lett. **63B**, 446 (1976).

[2] This paper generalizes results given in S. J. Aldersley, J. Math. Phys. **19**, 700 (1978).

[3] For further notation see S. J. Aldersley, Phys. Rev. **D15**, 370 (1977) (in particular, |i denotes the Levi-Civita covariant derivative).

[4] The number is based upon arguments similar to those given in chapter 7 of T. Y. Thomas, The Differential Invariants of Generalized Spaces, Cambridge Univ. Press, Cambridge 1934.

[5] See ref. [3] for an introduction to this technique and loc. cit. in ref. [2] and S. J. Aldersley, Gen. Rel. Grav. 8, 397 (1977), for analysis of similar problems.

[6] A special case of this approach is discussed in R. Skinner and D. Gregorash, Phys. Rev. **D14**, 3314, (1976). However, all terms involving the Levi-Civita tensor density (explicitly) are absent in this analysis.

[7] See the remark made following Theorem 2 of I. M. Anderson, Tensorial Euler-Lagrange Expressions and Conservation Laws, to appear in Aeq. Math.

[8] H. Weyl, The Classical Groups, Princeton Univ. Press, Princeton 1939.

[9] G. W. Horndeski, Utilitas Mathematica **9**, 3 (1976).